

On entropy function for supersymmetric black rings

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ABSTRACT: The entropy function for five-dimensional supersymmetric black rings, which are solutions of $U(1)^3$ minimal supergravity, is calculated via both on-shell and off-shell formalism. We find that at the tree level, the entropy function obtained from both perspectives can reproduce the Bekenstein-Hawking entropy. We also compute the higher order corrections to the entropy arising from five-dimensional Gauss-Bonnet term as well as supersymmetric R^2 completion respectively and compare the results with previous microscopic calculations.

KEYWORDS: Black Holes in String Theory, Black Holes.

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1. Introduction

The attractor mechanism has played an important role in understanding black hole physics in string theory and has been studied extensively in the past decade. It was initiated in the context of $N = 2$ extremal black holes [1] and generalized to more general cases, such as supersymmetric black holes with higher order corrections [2] and non-supersymmetric attractors [3–5].

Recently, based on Wald’s entropy formula [6], A.Sen proposed an effective method for calculating the entropy of D-dimensional black holes with near horizon geometry $AdS_2 \times S^{D-2}$, which is named as “entropy function” method [7]. It states that the entropy of such kind of black holes can be obtained by extremizing the “entropy function” with respect to various moduli, where the entropy function is defined as integrating the Lagrangian over the horizon coordinates and taking the Legendre transformation with respect to the electric charges. This method has been applied to many specific examples, such as extremal black holes in higher dimensions, rotating black holes and non-supersymmetric black holes. For recent developments, see [8]. It is also an useful way to calculate the higher order corrections to the entropy. In particular, more recently we have shown that for some nonextremal black holes in string theory, the entropy function method also works quite well [9]. Similar arguments that concerning entropy function for non-extremal black holes/branes appeared very recently [10].

It is well known that for black holes in four-dimensional asymptotically flat spacetime, there exists only one horizon topology S^2 . But for black holes in five-dimensional spacetime,

the horizon topology is not unique. A black hole solution with horizon topology $S^1 \times S^2$, named as black ring, was presented firstly in [11]. Several important developments are listed in [12–16], where various solutions, the microscopic entropy and relations to other topics are discussed. For reviews, see [17].

The near horizon geometry of certain black holes and black rings turns out to be $AdS_3 \times S^2$. It becomes $AdS_2 \times S^2$ after dimensional reduction so that one expects the entropy function formalism also works. Such attempts have been discussed in [18, 21] and an interesting paper appeared very recently [22], where the entropy function for five-dimensional extremal black holes and black rings is constructed in the context of two-derivative gravity coupled to abelian gauge fields and neutral scalar fields, which shows the validity of the entropy function in a general way.

Since the entropy function method can give the higher order corrections to the entropy conveniently, it is interesting to study the corrections via the entropy function and compare the results with microscopic calculations. In order to study the higher order corrections to the entropy, we take supersymmetric black rings in $U(1)^3$ supergravity as a concrete example. Firstly we carry out the analysis using the two-derivative on-shell and off-shell supergravity, where we make a dimensional reduction so that the resulting action is gauge and coordinate invariant. We find that the entropy function can reproduce both the Bekenstein-Hawking entropy and the near horizon geometry, while the correct attractor values of the moduli fields can also be obtained by extremizing the entropy function. Then we calculate the R^2 corrections to black ring entropy by adding the five-dimensional Gauss-Bonnet term which originates from R^4 terms in M-theory compactified on a Calabi-Yau manifold as well as supersymmetric R^2 completion and compare our results with previous microscopic considerations.

The rest of the paper is organized as follows. In section 2 we review the supersymmetric black ring solutions in the $U(1)^N$ theory and specialize to the case of $N = 3$. After dimensional reduction to four-dimensional spacetime, the entropy function for $U(1)^3$ black rings is carried out in section 3. The higher order corrections to the entropy are discussed in section 4. We summarize the results and discuss some related topics in section 5.

2. Supersymmetric black rings in $U(1)^3$ theory

In this section, we review some salient properties of supersymmetric black ring in the $U(1)^3$ theory, which are needed in the following calculations. For more details, see section II and appendix B of [13].

Consider the case of minimal supergravity coupled to $N - 1$ Abelian vector multiplets with scalars taking values in a symmetric space. The action for such a theory is

$$S = \frac{1}{16\pi G_5} \int \left(R \star 1 - G_{IJ} dX^I \wedge \star dX^J - G_{IJ} F^I \wedge \star F^J - \frac{1}{6} C_{IJK} A^I \wedge F^J \wedge F^K \right), \quad (2.1)$$

where $I, J, K = 1, \dots, N$ and the constants C_{IJK} are symmetric in (IJK) . The $N - 1$ dimensional scalar manifold is conveniently parameterized by the N scalars X^I , which obey the constraint

$$\frac{1}{6} C_{IJK} X^I X^J X^K = 1. \quad (2.2)$$

The matrix G_{IJ} is defined by

$$G_{IJ} \equiv \frac{9}{2}X_I X_J - \frac{1}{2}C_{IJK}X^K, \quad (2.3)$$

where $X_I \equiv \frac{1}{6}C_{IJK}X^J X^K$ such that $X_I X^I = 1$.

The supersymmetric black ring in $U(1)^3$ theory can be viewed as an eleven-dimensional supertube carrying three charges and three dipoles after dimensional reduction down to $D = 5$ on T^6 . The configuration can be summarized as follows:

$$\begin{array}{ll} Q_1 & M2 : 1 \ 2 \ - \ - \ - \ - \ - \ - , \\ Q_2 & M2 : - \ - \ 3 \ 4 \ - \ - \ - \ - , \\ Q_3 & M2 : - \ - \ - \ - \ 5 \ 6 \ - \ - , \\ p_1 & m5 : - \ - \ 3 \ 4 \ 5 \ 6 \ \psi , \\ p_2 & m5 : 1 \ 2 \ - \ - \ 5 \ 6 \ \psi , \\ p_3 & m5 : 1 \ 2 \ 3 \ 4 \ - \ - \ \psi . \end{array} \quad (2.4)$$

Such a configuration can be taken as a solution of $D = 11$ supergravity with the effective action

$$S_{11} = \frac{1}{16\pi G_{11}} \int \left(R_{11} \star_{11} 1 - \frac{1}{2} \mathcal{F} \wedge \star_{11} \mathcal{F} - \frac{1}{6} \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A} \right). \quad (2.5)$$

The eleven-dimensional solution describing this system takes the form

$$\begin{aligned} ds_{11}^2 &= ds_5^2 + X^1(dz_1^2 + dz_2^2) + X^2(dz_3^2 + dz_4^2) + X^3(dz_5^2 + dz_6^2), \\ \mathcal{A} &= A^1 \wedge dz_1 \wedge dz_2 + A^2 \wedge dz_3 \wedge dz_4 + A^3 \wedge dz_5 \wedge dz_6, \end{aligned} \quad (2.6)$$

where z_i denote the coordinates along the 123456-directions and \mathcal{A} is the three-form potential.

Note that if we reduce the eleven-dimensional action to five-dimensional spacetime on T^6 using the ansatz 2.6, we will obtain precisely the action 2.1 with $N = 3$, $C_{IJK} = 1$ if (IJK) is a permutation of (123) and $C_{IJK} = 0$ otherwise, and

$$G_{IJ} = \frac{1}{2} \text{diag}[(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}]. \quad (2.7)$$

The resulting five-dimensional black ring solution is characterized by the metric ds_5^2 , three scalars X^i , and three one-forms A^i , with field strengths $F^i = dA^i$, which are given by

$$\begin{aligned} ds_5^2 &= -(H_1 H_2 H_3)^{-2/3} (dt + \omega)^2 + (H_1 H_2 H_3)^{1/3} dx_4^2, \\ A^i &= H_i^{-1} (dt + \omega) - \frac{p_i}{2} [(1 + y) d\psi + (1 + x) d\phi], \\ X^i &= H_i^{-1} (H_1 H_2 H_3)^{1/3}, \end{aligned} \quad (2.8)$$

where

$$dx_4^2 = \frac{R^2}{(x-y)^2} \left[\frac{dy^2}{y^2-1} + (y^2-1)d\psi^2 + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2 \right], \quad (2.9)$$

$$H_1 = 1 + \frac{Q_1 - p_2 p_3}{2R^2} (x-y) - \frac{p_2 p_3}{4R^2} (x^2 - y^2),$$

$$H_2 = 1 + \frac{Q_2 - p_1 p_3}{2R^2} (x-y) - \frac{p_1 p_3}{4R^2} (x^2 - y^2), \quad (2.10)$$

$$H_3 = 1 + \frac{Q_3 - p_1 p_2}{2R^2} (x-y) - \frac{p_1 p_2}{4R^2} (x^2 - y^2), \quad (2.11)$$

and $\omega = \omega_\phi d\phi + \omega_\psi d\psi$ with

$$\omega_\phi = -\frac{1}{8R^2} (1-x^2) [p_1 Q_1 + p_2 Q_2 + p_3 Q_3 - p_1 p_2 p_3 (3+x+y)], \quad (2.12)$$

$$\omega_\psi = \frac{1}{2} (p_1 + p_2 + p_3) (1+y) - \frac{1}{8R^2} (y^2-1) [p_1 Q_1 + p_2 Q_2 + p_3 Q_3 - p_1 p_2 p_3 (3+x+y)].$$

Note that the six-torus T^6 has constant volume because $X^1 X^2 X^3 = 1$.

The horizon locates at $y = -\infty$ and in order to obtain the near horizon geometry, we have to take rather complicated coordinate transformations, which are discussed extensively in appendix D of [13] and here we will not repeat any more. The resulting near horizon metric is

$$ds^2 = \frac{4L}{p} \tilde{r} d\tilde{t} d\tilde{\psi} + L^2 d\tilde{\psi}^2 + \frac{p^2}{4} \frac{d\tilde{r}^2}{\tilde{r}^2} + \frac{p^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.13)$$

where

$$L \equiv \frac{1}{2p^2} \left[2 \sum_{i<j} Q_i p_i Q_j p_j - \sum_i Q_i^2 p_i^2 - 4R^2 p^3 \sum_i p_i \right],$$

$$Q_1 = Q_1 - p_2 p_3, \quad Q_2 = Q_2 - p_1 p_3, \quad Q_3 = Q_3 - p_1 p_2,$$

$$p \equiv (p_1 p_2 p_3)^{1/3}. \quad (2.14)$$

Finally, let $\tilde{t} = p^2 \tau / 4$ and $e^0 = p / 2L$, the near horizon metric (2.13) becomes

$$ds^2 = \frac{p^2}{4} (-\tilde{r}^2 d\tau^2 + \frac{d\tilde{r}^2}{\tilde{r}^2}) + L^2 (d\tilde{\psi} + e^0 \tilde{r} d\tau)^2 + \frac{p^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.15)$$

which is the product of a locally AdS_3 with radius p and a two-sphere of radius $p/2$. The Bekenstein-Hawking entropy is

$$S_{\text{BH}} = \frac{\mathcal{A}_5}{4G_5} = \frac{2\pi^2 L p^2}{4G_5}. \quad (2.16)$$

3. The entropy function analysis in four-dimensional spacetime

In this section, we will carry out the analysis of entropy function for supersymmetric black rings in detail, using both the on-shell and off-shell Lagrangian at two-derivative level.

3.1 On-shell analysis

In this subsection, we calculate the entropy of supersymmetric black rings in $U(1)^3$ supergravity via the entropy function formalism, which can be seen as a concrete example of [22]. According to [7], in order to carry out entropy function analysis, the Lagrangian must be gauge and coordinate invariant. Since the five-dimensional effective action contains a Chern-Simons term, the Lagrangian is not gauge invariant and we have to reduce it to four dimensions so that the entropy function can be applied. Such analysis was initiated in [19] and has been followed in [20, 21] and [22].

We take the near horizon field configuration as follows:

$$\begin{aligned}
 ds_5^2 &= w^{-1}[v_1(-r^2 dt^2 + dr^2/r^2) + v_2(d\theta^2 + \sin^2 \theta d\phi^2)] + w^2(d\psi + e^0 r dt)^2, \\
 A_5^I &= A_4^I + a^I(d\psi + e^0 r dt), \\
 F_{5rt}^I &= e^I + a^I e^0, \quad F_{4rt}^I = e^I, \quad F_{4rt}^0 = e^0, \quad F_{5\theta\phi}^I = F_{4\theta\phi}^I = \frac{1}{2} p^I \sin \theta, \\
 X^I &= x^I, \quad I = 1, 2, 3.
 \end{aligned} \tag{3.1}$$

Note that the ψ components of the gauge potential become axions in four-dimensional spacetime. The entropy function analysis will be processed in four-dimensional spacetime after dimensional reduction on ψ coordinate. Define

$$f_0 \equiv \frac{1}{16\pi} \int d\theta d\phi \sqrt{-g} (\mathcal{L}'_0 + \mathcal{L}_{0CS}) = f'_0 + f_{0CS}, \tag{3.2}$$

where \mathcal{L}_{0CS} is the Chern-Simons term down to four dimensions and \mathcal{L}'_0 denotes the resulting Lagrangian coming from the gauge and coordinate invariant terms in the original five-dimensional supergravity. Throughout the work, the four-dimensional Newton constant is set to be $G_4 \equiv 1$ so that $G_5 = 2\pi$.

Note that for a consistent dimensional reduction, the first term in (3.2) can be evaluated in the original five-dimensional background without writing out the expression for the reduced action explicitly, what we should pay attention to is the second Chern-Simons term. In four-dimensional spacetime such a term becomes

$$\begin{aligned}
 \frac{1}{6} A_5 \wedge F_5 \wedge F_5 &= e^{-1} \left(\frac{1}{6} C_{IJK} a^I a^J a^K F_{4\mu\nu}^0 F_{4\lambda\sigma}^0 \epsilon^{\mu\nu\lambda\sigma} + \frac{1}{4} C_{IJK} a^J a^K F_{4\mu\nu}^I F_{4\lambda\sigma}^0 \epsilon^{\mu\nu\lambda\sigma} \right. \\
 &\quad \left. + \frac{1}{4} C_{IJK} a^J a^K F_{4\mu\nu}^0 F_{4\lambda\sigma}^I \epsilon^{\mu\nu\lambda\sigma} + \frac{1}{2} C_{IJK} a^K F_{4\mu\nu}^I F_{4\lambda\sigma}^J \epsilon^{\mu\nu\lambda\sigma} \right), \\
 \mu, \nu, \lambda, \sigma &= t, r, \theta, \phi, \quad I = 1, 2, 3.
 \end{aligned} \tag{3.3}$$

Next, define

$$F_0 \equiv e^0 \frac{\partial f_0}{\partial e^0} + e^I \frac{\partial f_0}{\partial e^I} - f_0, \tag{3.4}$$

where F_0 is a function of v_1, v_2, w, a^I and x^I . Finally, the entropy is given by

$$S_{\text{BR}} = 2\pi F_0 \tag{3.5}$$

after extremizing F_0 with respect to various moduli and substituting their values back into F_0 .

We can obtain the explicit results of \mathcal{L}'_0 and \mathcal{L}_{0CS} directly by putting the near horizon field configuration (3.1) into (3.2),

$$\begin{aligned}\mathcal{L}'_0 &= \left(-\frac{2w}{v_1} + \frac{2w}{v_2} + \frac{(e^0)^2 w^4}{2v_1^2} \right) + \frac{1}{2}(x^I)^{-2} \frac{w^2}{v_1^2} (e^I + a^I e^0)^2 - \frac{1}{8}(x^I)^{-2} \frac{w^2}{v_2^2} (p^I)^2, \\ \mathcal{L}_{0CS} &= 2e^{-1} \sin \theta C_{IJK} (e^0 p^I a^J a^K + e^I p^J a^K).\end{aligned}\tag{3.6}$$

One subtle is that the definition of electric charges will receive modifications in the presence of Chern-Simons terms. In the usual analysis of entropy function, the electric charges are defined by $q_I \equiv \partial f / \partial e^I$. However, when Chern-Simons terms are taken into account, the corresponding expression gives the so-called ‘‘Page charge’’ introduced in [23], whose definition is given by

$$Q_{\text{Page}} \sim \int *F + A \wedge F.\tag{3.7}$$

The subsequent calculations are straightforward and the expression for F_0 is

$$F_0 = \frac{v_2}{2} - \frac{v_1}{2} + \frac{p^2 L^4 v_1}{32 v_2 w^3} + \frac{1}{8} (x^I)^{-2} \frac{w^2}{v_1^2} (e^I + a^I e^0)^2 + \frac{1}{32} (x^I)^{-2} \frac{w^2}{v_2^2} (p^I)^2,\tag{3.8}$$

where we have replaced e^0 by the ‘‘true’’ electric charge $q_0 \equiv \partial f'_0 / \partial e^0 = \frac{1}{4} v_1^{-1} v_2 w^3 e^0$. We can obtain the correct values of the various moduli fields by solving the following equations

$$\frac{\partial F_0}{\partial v_1} = \frac{\partial F_0}{\partial v_2} = \frac{\partial F_0}{\partial w} = 0, \quad \frac{\partial F_0}{\partial x^I} = \frac{\partial F_0}{\partial a^I} = 0.\tag{3.9}$$

The solutions to the above equations are given as follows

$$v_1 = v_2 = \frac{L p^2}{4}, \quad w = L, \quad x^I = \frac{p^I}{p}, \quad a^I = -\frac{e^I}{e^0},\tag{3.10}$$

note that the x^I s are not independent, subject to the constraint $x^1 x^2 x^3 = 1$. Thus we have obtained the correct near horizon geometry and attractor values of the scalar fields. Furthermore, we can obtain the entropy by putting all the values back into F ,

$$S_{\text{BR}} = 2\pi F_0 = \frac{\pi L p^2}{4},\tag{3.11}$$

which reproduces the Bekenstein-Hawking entropy.

3.2 Off-shell analysis

We will calculate the entropy function using the off-shell formalism. One advantage of the off-shell formalism is that the supersymmetric completion of an R^2 term can be realized more conveniently. For simplicity, we just list the basic ingredients of the relevant supermultiplets briefly. Details for the five-dimensional off-shell supergravity can be found in [26] and references therein. Similar work has been done in [29], where both the entropy function formalism and the so-called ‘‘c-extremization’’ [30] are discussed.

The irreducible Weyl multiplet, which consists of 32 bosonic plus 32 fermionic component fields, contains the following fields

$$e_\mu^a, \quad \psi_\mu^i, \quad V_\mu^{ij}, \quad b_\mu, \quad v^{ab}, \quad \chi^i, \quad D, \quad (3.12)$$

where e_μ^a are the vielbein, V_μ^{ij} and b_μ denote gauge fields associated with the SU(2) generator and dilatation generator respectively. ψ_μ^i and χ^i are SU(2)-Majorana spinors. Note that v^{ab} , χ^i and D are auxiliary fields, where v^{ab} is antisymmetric in a and b and D is a scalar. The vector multiplet consists of gauge fields A_μ^I , scalar fields M^I , SU(2)-Majorana gaugini Ω^I and SU(2)-triplet auxiliary fields Y^{IJ} , which can be gauged away.

After gauge fixing, the bosonic terms in the two-derivative Lagrangian of $\mathcal{N} = 2$ supergravity with the Weyl multiplet and n_v vector multiplets can be expressed as

$$\begin{aligned} \mathcal{L}_0 = & -\frac{1}{2}D + \frac{3}{4}R + v^2 + \mathcal{N} \left(\frac{1}{2}D + \frac{1}{4}R + 3v^2 \right) + 2\mathcal{N}_I v^{ab} F_{ab}^I \\ & + \mathcal{N}_{IJ} \left(\frac{1}{4}F_{ab}^I F^{Jab} + \frac{1}{2}\partial_a M^I \partial^a M^J \right) + \frac{1}{24}e^{-1}C_{IJK}A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde}, \end{aligned} \quad (3.13)$$

where the functions characterizing the scalar manifold are defined as

$$\mathcal{N} = \frac{1}{6}C_{IJK}M^I M^J M^K, \quad \mathcal{N}_I = \partial_I \mathcal{N} = \frac{1}{2}C_{IJK}M^J M^K, \quad \mathcal{N}_{IJ} = C_{IJK}M^K, \quad (3.14)$$

with $I, J, K = 1, \dots, n_v$. Note that the equation of motion for the auxiliary field D fixes $\mathcal{N} = 1$, that is, the scalars parametrize the ‘‘very special geometry’’. The auxiliary fields v^{ab} and D can be eliminated via their equations of motion and the resulting Lagrangian is the familiar one arising from the compactification of eleven-dimensional supergravity on a Calabi-Yau manifold with intersection numbers C_{IJK} .

The near horizon field configuration can be taken as follows

$$\begin{aligned} ds^2 = & w^{-1}[v_1(-r^2 dt^2 + dr^2/r^2) + v_2(d\theta^2 + \sin^2 \theta d\phi^2)] + w^2(d\psi + e^0 r dt)^2, \\ F_{5rt}^I = & e^I + e^0 a^I, \quad F_{4rt}^I = e^I, \quad F_{4rt}^0 = e^0 = \frac{p}{2L}, \quad F_{5\theta\phi}^I = F_{4\theta\phi}^I = \frac{1}{2}p^I \sin \theta, \\ M^I = & mp^I, \quad v_{\theta\phi} = V \sin \theta. \end{aligned} \quad (3.15)$$

The auxiliary fields can be eliminated by solving the equations

$$\frac{\partial \mathcal{L}_0}{\partial D} = 0, \quad \frac{\partial \mathcal{L}_0}{\partial V} = 0, \quad \frac{\partial \mathcal{L}_0}{\partial m} = 0, \quad (3.16)$$

which gives

$$D = 12p^{-2}, \quad m = p^{-1}, \quad V = -\frac{3}{8}p. \quad (3.17)$$

After substituting (3.17) back in to \mathcal{L}_0 , which gives

$$\begin{aligned} \mathcal{L}_0 = & \left(-\frac{2w}{v_1} + \frac{2w}{v_2} + \frac{(e^0)^2 w^4}{2v_1^2} \right) - \frac{3w^2}{8v_2^2} p^2 - \frac{1}{2}C_{IJK}(e^I + e^0 a^I)(e^J + e^0 a^J) \frac{p^K}{p} \frac{w^2}{v_1^2} \\ & - 2e^{-1} \sin \theta C_{IJK}(e^0 p^I a^J a^K + e^I p^J a^K). \end{aligned} \quad (3.18)$$

Then the subsequent analysis is similar to the on-shell case discussed in the previous subsection and the same result will be obtained, which would not be repeated here.

4. Higher order corrections

In this section we would like to discuss higher order corrections to black ring entropy, which can be obtained in a similar way by incorporating the higher order corrections into the effective action. We will use two different actions, one of which is the five-dimensional Gauss-Bonnet term coming from the compactification of M-theory on a Calabi-Yau three fold CY_3 [25], while the other is a supersymmetric completion of R^2 terms in five-dimensional supergravity proposed recently [26]. We also compare our results with the one obtained from microscopic considerations.

4.1 5D Gauss-Bonnet corrections

The higher order corrections to the low energy effective action for the compactification of M-theory on a Calabi-Yau threefold CY_3 (here is T^6) down to five dimensions takes the following form

$$I_{\text{GB}} = \frac{1}{2^9 3 \pi^2} \int d^5 x \sqrt{-g} c \cdot X (R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2), \quad (4.1)$$

where $c \cdot X = c_{2I} X^I$ with c_{2I} denoting the components of the second class of CY_3 . In principle, we should do dimensional reduction on \mathcal{L}_{GB} down to four-dimensional spacetime. However, as pointed in previous section, such a term can be evaluated in the original five-dimensional background without the explicit resulting four-dimensional action, for this term is gauge and coordinate invariant.

Define

$$f_1 \equiv \frac{1}{16\pi} \int d\theta d\phi \sqrt{-g} \mathcal{L}_{\text{GB}} \quad (4.2)$$

and substitute the five-dimensional near horizon metric in (3.1) into the above expression, we can arrive at the following result

$$f_1 = \frac{1}{3 \cdot 2^6} \frac{c_{2I} p^I}{p} \left(-8w + \frac{2(e^0)^2 w^4}{v_1} \right) \quad (4.3)$$

where the scalar fields X^I have been taken their attractor values $X^I = p^I/p$. Then we can redefine $F \equiv F_0 - f_1$ in a straightforward way and obtain the corrections to the entropy. However, it seems that such equations can not give explicit solutions for the moduli fields.

As pointed out in [20], since the entropy is related to the value of F at its extremum, a first order error in the determination of the near horizon background will give a second order error in the value of the entropy. Thus if we want to obtain the first order correction to the entropy, we can find the near horizon background just by extremizing F_0 and then evaluate F in this background. Finally, the entropy is given by

$$S_{\text{BH}} = 2\pi F, \quad (4.4)$$

after substituting the values of various moduli fields obtained by extremizing F_0 . Thus we can obtain the first order corrections to black ring entropy

$$\Delta S_{\text{BR}} = -2\pi f_1 = \frac{\pi}{16} c \cdot p \frac{L}{p}. \quad (4.5)$$

4.2 Supersymmetric R^2 corrections

One shortcoming of the five-dimensional Gauss-Bonnet Lagrangian is that it is not supersymmetric, so several terms relevant to the entropy might be omitted. Fortunately, a supersymmetric completion of R^2 terms in five-dimensional supergravity has been realized recently in [26] and several relevant applications have been listed in [29]. Thus one can revisit the higher order corrections by making use of the supersymmetric R^2 action.

A particular higher order term has been taken into account in [26] and the supersymmetric completion has been realized there. The particular term is the so-called mixed gauge-gravitational Chern-Simons term

$$e\mathcal{L}_{\text{CS}} = -\frac{c_{2I}}{6 \cdot 2} A^I \wedge \text{Tr}(R \wedge R) = \frac{c_{2I}}{3 \cdot 16} \epsilon_{abcde} A^{Ia} R^{bcfg} R^{de}{}_{fg}, \quad (4.6)$$

with c_{2I} being the expansion coefficients of the second Chern class, which comes from the anomaly arguments. The four derivative supersymmetric completion of the Chern-Simons term has been given in [26] and the relevant terms for the calculation are¹

$$\begin{aligned} \mathcal{L}_1 = \frac{c_{2I}}{3} & \left(\frac{1}{16} e^{-1} \epsilon_{abcde} A^{Ia} C^{bcfg} C^{de}{}_{fg} + \frac{1}{8} M^I C^{abcd} C_{abcd} + \frac{1}{12} M^I D^2 + \frac{1}{6} F^{Iab} v_{ab} D \right. \\ & - \frac{1}{3} M^I C_{abcd} v^{ab} v^{cd} - \frac{1}{2} F^{Iab} C_{abcd} v^{cd} + \frac{8}{3} M^I v_{ab} \hat{\mathcal{D}}^b \hat{\mathcal{D}}^c v^{ac} \\ & + \frac{4}{3} M^I \hat{\mathcal{D}}^a v^{bc} \hat{\mathcal{D}}_a v_{bc} + \frac{4}{3} M^I \hat{\mathcal{D}}^a v^{bc} \hat{\mathcal{D}}_b v_{ca} - \frac{2}{3} e^{-1} M^I \epsilon_{abcde} v^{ab} v^{cd} \hat{\mathcal{D}}_f v^{ef} \\ & + \frac{2}{3} e^{-1} F^{Iab} \epsilon_{abcde} v^{cd} \hat{\mathcal{D}}_f v^{ef} + e^{-1} F^{Iab} \epsilon_{abcde} v^c{}_f \hat{\mathcal{D}}^d v^{ef} \\ & \left. - \frac{4}{3} F^{Iab} v_{ac} v^{cd} v_{db} - \frac{1}{3} F^{Iab} v_{ab} v^2 + 4M^I v_{ab} v^{bc} v_{cd} v^{da} - M^I (v_{ab} v^{ab})^2 \right), \quad (4.7) \end{aligned}$$

with C_{abcd} the Weyl tensor defined as

$$C_{abcd} = R_{abcd} - \frac{2}{3} (g_{a[c} R_{d]b} - g_{b[c} R_{d]a}) + \frac{1}{6} R g_{a[c} g_{d]b}. \quad (4.8)$$

The double covariant derivative of v_{ab} is given by

$$v_{ab} \hat{\mathcal{D}}^b \hat{\mathcal{D}}^c v^{ac} = v_{ab} \mathcal{D}^b \mathcal{D}_c v^{ac} + \frac{2}{3} v^{ac} v_{cb} R_a{}^b + \frac{1}{12} v_{ab} v^{ab} R, \quad (4.9)$$

where the superconformal derivative is related to the usual derivative as $\hat{\mathcal{D}}_\mu = \mathcal{D}_\mu - b_\mu$.

For our background which satisfies $\hat{\mathcal{D}}_a v_{bc} = 0$, following [20], the first order corrections to the black ring entropy can be obtained by substituting (3.10) and (3.17) into

$$\Delta' S_{\text{BR}} = 2\pi F'_1 = -2\pi f'_1, \quad (4.10)$$

where

$$f'_1 \equiv \frac{1}{16\pi} \int d\theta d\phi \sqrt{-g} \mathcal{L}_1. \quad (4.11)$$

¹The difference in the overall coefficients is due to the different conventions of the five-dimensional Newton constant.

The result is

$$\Delta' S_{\text{BR}} = \frac{\pi c \cdot p}{8 p} L, \quad (4.12)$$

which shows similar behavior as the Gauss-Bonnet corrections but with different numerical coefficients.

4.3 Comparison with the microscopic corrections

Now we would like to compare our results with the microscopic entropy obtained in previous papers [27, 28]. The microscopic corrections reads

$$\Delta S_{\text{BRmic}} = \frac{\pi}{6} c_2 \cdot p \sqrt{\frac{\hat{q}_0}{C}}, \quad (4.13)$$

where \hat{q}_0 denotes the left-moving oscillator number and C is related to the left-moving central charge c_L by $c_L = 6C$. The expressions for the quantities are given as follows

$$\begin{aligned} \hat{q}_0 &= -J_\psi + \frac{1}{12} C^{AB} q_A q_B + \frac{c_L}{24}, \\ &= -J_\psi + \frac{C}{4} + \frac{1}{16} \left(\frac{q_1 q_2}{p^3} + \frac{q_2 q_3}{p^1} + \frac{q_1 q_3}{p^2} \right) - \frac{1}{256C} ((p^1 q_1)^2 + (p^2 q_2)^2 + (p^3 q_3)^2), \\ C^{AB} &= (C_{ABC} p^C)^{-1} = (C_{ABC} p^A p^B p^C)^{-1}, \\ c_L = 6C &= 6C_{ABC} p^A p^B p^C = \frac{3}{4} p^1 p^2 p^3. \end{aligned} \quad (4.14)$$

One can see that

$$\hat{q}_0 = \frac{1}{8} L^2 p, \quad C = \frac{1}{8} p^3, \quad (4.15)$$

then

$$\Delta S_{\text{BRmic}} = \frac{\pi}{6} c \cdot p \frac{L}{p}, \quad (4.16)$$

Comparing (4.5) and (4.12) with the above microscopic correction, we can find that all of them behave similarly up to a numerical constant. Furthermore, the correction obtained from the supersymmetric completion is more accurate than the Gauss-Bonnet correction, which means that the former action contains more terms that are necessary in considering the higher order corrections to black ring solutions.

5. Summary and discussion

In this paper, we have constructed the entropy function for supersymmetric black rings in $U(1)^3$ theory from both on-shell and off-shell perspectives, which can be seen as a concrete realization of [22]. We have found that after dimensional reduction down to four-dimensional spacetime, the effective action is gauge invariant and we can carry out the entropy function analysis. We have reproduced the Bekenstein-Hawking entropy precisely and have obtained the correct attractor values of the scalar fields via the entropy function using both formalisms. Note that when we set $Q_1 = Q_2 = Q_3$, $q_1 = q_2 = q_3$, the black ring

²Note that $q_{\text{there}} = \frac{1}{4} q_{\text{here}}$, $p_{\text{there}}^I = \frac{1}{2} p_{\text{here}}^I$

becomes the solution presented in [12], which implies that the entropy function formalism can also be applied to those cases.

The higher order corrections to the entropy have also been discussed and two kinds of corrections have been considered. One is due to the five-dimensional Gauss-Bonnet term which comes from R^4 corrections in eleven-dimensional M-theory compactified on CY_3 but is not supersymmetric itself, while the other arises from a supersymmetric R^2 completion of the five-dimensional supergravity. Unfortunately, we have found that although both of them can give similar behavior as the microscopic result, the numerical coefficients can not be reproduced successfully. However, the result obtained via the supersymmetric R^2 completion is more accurate than that obtained via the five-dimensional Gauss-Bonnet term, which means that the former contains more information about the corrections and there might still be some relevant terms ignored.

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